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Tails in Harnesses

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We consider space- and time-uniform d-dimensional random processes with linear local interaction, which we call harnesses and which may be used as discrete mathematical models of random interfaces. Their components are real random variables a_s^t, where $s \in \mathbf{Z}^d$ and $t = 0, 1, 2,...$. At every time step two events occur: first, every component turns into a linear combination of its N neighbors, and second, a symmetric random i.i.d. "noise" v is added to every component. For any $\sigma \in \mathbf{Z}_+^d$ define $\Delta_\sigma a_s^t$ as follows. If $\sigma = (0,..., 0)$, $\Delta_\sigma a_s^t = a_s^t$. Then by induction, $\Delta_{\sigma + e_i} a_s^t = \Delta_\sigma a_{s+e_i}^t - \Delta_\sigma a_s^t$, where e_i is the d-dimensional vector, whose ith component is one and other components are zeros. Denote $|\sigma|$ the sum of components of σ. Call a real random variable ξ symmetric if it is distributed as $-\xi$. For any symmetric random variable ξ *power decay* or P-decay is defined as the supremum of those r for which the rth absolute moment of ξ is finite. Convergence a.s., in probability and in law when $t \to \infty$ is examined in terms of P-decay(v): If $d = 1$, $\sigma = 0$ or $d = 2$, $\sigma = (0, 0)$, $\Delta_\sigma a_s^t$ diverges. In all the other cases: If P-decay(v) $< (d+2)/(d + |\sigma|)$, $\Delta_\sigma a_s^t$ diverges; if P-decay(v) $> (d+2)/(d+|\sigma|)$, $\Delta_\sigma a_s^t$ converges and P-decay(lim $\Delta_\sigma a_s^t$) = P-decay(v). For any symmetric random variable ξ *exponential decay* or E-decay is defined as the supremum of those r for which the expectation of $\exp(|x|^r)$ is finite. Let E-decay(v) > 0. Whenever $\Delta_\sigma a_s^t$ converges (that is, if $d > 2$ or $|\sigma| > 0$): If $d > 2$, E-decay(lim a_s^t) = min(E-decay(v), $(d+2)/2$); if $|\sigma| = 1$, E-decay (lim $\Delta_\sigma a_s^t$) = min(E-decay(v), $d+2$); if $|\sigma| \geq 2$, E-decay(lim $\Delta_\sigma a_s^t$) = E-decay(v).

KEY WORDS: Harnesses; interacting random processes; dynamics of surfaces; interface roughening; decay of random distributions; tails of distributions; convergence of random series; random walks; Cramér–Edgeworth asymptotic expansions of lattice distributions; integral transforms.

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1. BACKGROUND, DEFINITIONS, AND FORMULATIONS

Some of the most important models of mathematical physics are not restricted to one area of physics; instead they provide mathematical apparatus which can be used in different areas. We suggest that *harnesses* considered here share this property. Harnesses (defined below) are a generalization of "one-sided harnesses" introduced by Hammersley.[6] His motivation was to study and explain long-range correlation between sub-grains of metals. Without this correlation no crystallic structure would be possible. The Edwards–Wilkinson equation[4] is a continuous analog of harnesses, although its primary physical counterpart was quite different: it was surface fluctuations in a settled granular material. Thus we conclude that harnesses deserve to be studied in general.

Components of our processes are real random variables, or r.v. for short, which are indexed by d-dimensional vectors with integer components, denoted $s \in \mathbf{Z}^d$. Choose a natural number $N \geqslant d+1$ and N different d-dimensional vectors $v_1, ..., v_N$ with integer components, whose differences generate $\mathbf{Z}^d$. Components $s+v_1, ..., s+v_N$ are those which influence the component s at every step of the discrete time; we call them *neighbors* of s. Also choose intensities of these influences, that is, N positive numbers $w_1, ..., w_N$, whose sum equals 1. We define a *harness* as a joint distribution of r.v. a_s^t, where $s \in \mathbf{Z}^d$ and $t = 0, 1, 2, ...$, which is induced by the distribution of i.i.d. random variables v_s^t, every one of which is distributed as a given nonconstant symmetric r.v. v, which we call *noise*, with the map, defined in the following inductive way:

$$a_s^t = \sum_{i=1}^{N} w_i \cdot a_{s+v_i}^{t-1} + v_s^t \qquad \text{for all} \quad t = 1, 2, 3, ... \tag{1}$$

Here a_s^0 are components of the initial condition, which we assume to be zeros. Thus a harness is specified by a number N, by $v_1, ..., v_N$ and $w_1, ..., w_N$, which satisfy the mentioned conditions, and by a symmetric r.v. v. All the values which depend only on these parameters will be called constants.

Time and space in our processes are discrete, as in Chapter 9 of (11) and papers cited there. However, our models may be compared with those with continuous space and time, which are widely discussed in the physical literature (see e.g. refs. 1, 7, 9, 13, 14, and 17). The random noise v is symmetric, but otherwise arbitrary in this paper. We show that the tail of the distribution of noise influences the behavior of the system. Similar phenomena have been observed in the physical literature: "the influence of the *noise distribution* comes as a surprise, since such microscopic details are traditionally expected to be irrelevant for the large scale, long time properties."[9] "For the present growth problem I shall show that microscopic details can indeed influence large scale behavior in a substantial way, thus violating the naive universality concept."[17]

Derivatives of some physical quantities are also important physical quantities. In our case space and time are discrete, so we need to consider discrete analogs of derivatives: iterated differences. We denote them $\Delta_\sigma a_s^t$ for all $\sigma \in \mathbf{Z}_+^d$, where $\mathbf{Z}_+ = \{0, 1, 2,...\}$, and define them in the following way. If all components of σ are zeros, $\Delta_\sigma a_s^t = a_s^t$. After that define by induction for all σ, i, s, t

$$\Delta_{\sigma + e_i} a_s^t = \Delta_\sigma a_{s+e_i}^t - \Delta_\sigma a_s^t \tag{2}$$

where e_i is the d-dimensional vector, whose ith component equals one and all the other components are zeros. Note that this definition is consistent. Even when a_s^t diverge when $t \to \infty$, their differences $\Delta_\sigma a_s^t$ may well converge, as shown by our Theorem 1, and we believe that it is their convergence which makes it possible to use our models to describe such compact physical phenomena as interfaces. Indeed, when physicists speak about an "interface" they certainly mean something that can be localized and does not dissipate.

It seems clear that the first differences (those with $|\sigma| = 1$) are physically relevant. If a_s^t represent the orientation of subgrains in metals, as in ref. 6, absolute values of their first differences may influence the metal's strength: while first differences are small enough, the metal remains strong, even if a_s^t diverge. If a_s^t represent the height of a sandpile, as in ref. 4, their first differences are components of the gradient. We may expect the sandpile to be stable while the norm of the gradient remains small enough everywhere, even if the height diverges.[2] We do not discuss the physical relevance of higher differences (those with $|\sigma| > 1$), but hope to show in the future that some of them are also relevant.

Given a r.v. ξ, we denote $F(x) = F_\xi(x) = F(x \mid \xi) = \text{Prob}(\xi < x)$ its distribution function and $\bar{F}(x) = 1 - F(x)$. For any $r > 0$ denote $M_r(\xi)$ the rth absolute moment of ξ, that is, the expectation of $|\xi|^r$. If $r = 0$, we define $M_0(\xi) = 1$. We use two characteristics of how fast a symmetric random distribution decays at infinity: *P-decay* and *E-decay*. Given a symmetric r.v. ξ, call its *power decay*, or *P-decay* for short, the supremum of those r for which the rth absolute moment of ξ is finite. P-decay can equal any nonnegative number or infinity. For example, P-decay (ξ) equals $r > 0$ if $dF_\xi(x)/dx = \text{const}/(1 + |x|^{r+1})$. P-decay is used in Theorem 1, which gives

[2] See a discussion of avalanches in ref. 13. See also a discussion of the distinction between bounded and unbounded slopes in ref. 14. See also ref. 16, where $d = 1$ and the convergence and limit distribution of $\Delta_1 a_s^t$ are important.

criteria of convergence for harnesses, and in Theorem 2, which gives criteria of convergence for some random series.

Theorem 1. For any harness:

(a) If $d=1$, $\sigma=0$ or $d=2$, $\sigma=(0,0)$, a_s^t diverges when $t\to\infty$.[3]

(b) In all the other cases, that is, if $d>2$ or $|\sigma|>0$:

- If P-decay$(v)<(d+2)/(d+|\sigma|)$, $\Delta_\sigma a_s^t$ diverges when $t\to\infty$.
- If P-decay$(v)>(d+2)/(d+|\sigma|)$, $\Delta_\sigma a_s^t$ converges when $t\to\infty$ and P-decay$(\lim_{t\to\infty}\Delta_\sigma a_s^t)=$ P-decay(v).[4]

Here and in similar cases convergence and divergence are a.s., in probability and in law. $|\sigma|$ denotes the sum of components of σ. Theorem 1 follows from Theorem 2, which we are going to formulate.

Given a sequence $p_1, p_2,\ldots$ of nonnegative numbers which tends to zero, call $\mathrm{Deg}(p_k)$ or the *degree* of this sequence the supremum of those r for which the series $\sum_{k=1}^{\infty} p_k^r$ diverges. Note that degree of a sequence does not change when we permute its terms and delete zero terms. Using this, we may assume without any substantial loss of generality that $p_k\geqslant p_{k+1}>0$ for all k. Under this assumption

$$\mathrm{Deg}(p_k)=\limsup_{k\to\infty}\,(-\log_{p_k} k)=1/\liminf_{k\to\infty}\,(-\log_k p_k) \tag{3}$$

In this and similar cases we assume that $1/0=\infty$ and $1/\infty=0$.

Theorem 2. Given a nonconstant symmetric r.v. ξ and a sequence $p_1, p_2,\ldots$ of positive numbers which tends to 0, consider the random series

$$\theta=\sum_{k=1}^{\infty} p_k\cdot\xi_k \tag{4}$$

where ξ_k are independent r.v. distributed as ξ.

(a) If $\mathrm{Deg}(p_k)>2$ or $\mathrm{Deg}(p_k)>$ P-decay(ξ), the series (4) diverges.

(b) If $\mathrm{Deg}(p_k)<2$ and $\mathrm{Deg}(p_k)<$ P-decay(ξ), the series (4) converges and P-decay$(\theta)=$ P-decay(ξ).

[3] Both statements of (a) are well known and have been proved, e.g., in ref. 6. We include them for completeness and to show how easily they follow from our approach. About growth of variance of a_s^t see ref. 6. About growth of variance of $\Delta_\sigma a_s^t$ see Note 1 below.

[4] Based on physical considerations, J. Krug predicted both statements of (b) for the case $|\sigma|=0$ (private communication).

(c) If $\mathrm{Deg}(p_k) = 2 < \text{P-decay}(\xi)$, the series (4) converges if and only if the series $\sum_{k=1}^{\infty} p_k^2$ converges. If the series (4) converges in this case, then also $\text{P-decay}(\theta) = \text{P-decay}(\xi)$.

There are two intermediate cases: $\text{P-decay}(\xi) > \mathrm{Deg}(p_k) = 2$ and P-decay $(\xi) = \mathrm{Deg}(p_k) \leqslant 2$. The former is and the latter is not covered by Theorem 2. Examples 3 and 4 show that both convergence and divergence are possible in these cases.

Now we go to Theorems 3 and 4. Given a symmetric r.v. ξ, let us call its *exponential decay* or *E-decay* the supremum of those r for which the expectation of $\exp(|x|^r)$ is finite. It is easy to prove that

$$\text{E-decay}(\xi) = \liminf_{x \to \infty} \log_x(-\ln \mathrm{Prob}(\xi > x)) \tag{5}$$

E-decay can equal any nonnegative number or infinity. For example, $\text{E-decay}(\xi)$ equals $r > 0$ if $dF_\xi(x)/dx = \mathrm{const} \cdot \exp(-|x|^r)$. Theorem 3 shows that E-decay of the limit distribution of components of a harness and of their first differences may depend on the dimension of the harness.

Theorem 3. Assume that $\text{E-decay}(v) > 0$. Exclude the divergent cases $d = 1$, $\sigma = 0$ and $d = 2$, $\sigma = (0, 0)$. In all the other cases $\Delta_\sigma a_s^t$ converges a.s., in probability and in law when $t \to \infty$ and:

- If $d > 2$, $\text{E-decay}(\lim_{t \to \infty} a_s^t) = \min(\text{E-decay}(v), (d+2)/2)$.
- If $|\sigma| = 1$, $\text{E-decay}(\lim_{t \to \infty} \Delta_\sigma a_s^t) = \min(\text{E-decay}(v), d+2)$.
- If $|\sigma| \geqslant 2$, $\text{E-decay}(\lim_{t \to \infty} \Delta_\sigma a_s^t) = \text{E-decay}(v)$.

Theorem 3 is a direct corollary of the following Theorem 4 and Lemma 1.[5]

Theorem 4. Consider the random series

$$\theta = \sum_{k=1}^{\infty} p_k \cdot \xi_k \tag{6}$$

where ξ_k are independent r.v., each distributed as a given nonconstant symmetric r.v. ξ, such that $\text{E-decay}(\xi) > 0$. Also assume that the sequence $p_1, p_2, \ldots$ tends to 0 and $\mathrm{Deg}(p_k) = D < 2$. Then the series (6) converges and

$$\text{E-decay}(\theta) = \begin{cases} \dfrac{D}{D-1} & \text{if } D > 1 \text{ and } \text{E-decay}(\xi) > \dfrac{D}{D-1} \\ \text{E-decay}(\xi) & \text{otherwise} \end{cases} \tag{7}$$

[5] In the special case $d = \sigma = 1$ an analog of Theorem 3 was proved in ref. 16.

It is easy to prove that E-decay of a sum of several independent symmetric r.v. equals the minimum of their E-decays (see formula (25) below). Formula (7) shows that for series this does not need to be true: E-decay of the sum of a series may be less than E-decay of every summand. In informal terms, formula (7) describes two different mechanisms of formation of a large value of the sum (6): it may be caused either by a large value of one summand (the second line), or by the accumulation of small values of many summands (the first line). There are analogous observations in the physical literature. For example, ref. 9 concentrates on "the *largest* noise fluctuations" motivating this by the observations that "the interface advances in occasional large thrusts which then rapidly spread in the lateral direction." This corresponds to the second line of (7). How does interface behave if the first line takes place? It would be interesting to illustrate the difference between the two modes of behavior by computer simulation.

2. PROOF OF THEOREM 1

Let us show how Theorem 1 follows from Theorem 2. Since our processes are space-uniform, distributions of a_s^t do not depend on s, so we may concentrate on the case $s=0$. Since the initial conditions are zeros, a_0^t are linear combinations of some v_s^t:

$$a_0^t = \sum_{n=0}^{t-1} \sum_s p_n(s) \cdot v_s^{t-n} \tag{8}$$

where $p_n(s)$ denote the coefficients. Hence the iterated differences $\Delta_\sigma a_s^t$ are linear combinations of some v_s^t

$$\Delta_\sigma a_0^t = \sum_{n=0}^{t-1} \sum_s \Delta_\sigma p_n(s) \cdot v_s^{t-n} \tag{9}$$

where the coefficients $\Delta_\sigma p_n(s)$ are iterated differences of $p_n(s)$. Now we can use the time-uniformity of our systems to assume that the process starts at $-t$ and rewrite (9) as

$$\Delta_\sigma a_0^0 = \sum_{n=0}^{t-1} \sum_s \Delta_\sigma p_n(s) \cdot v_s^{-n}$$

where v_s^{-n} are i.i.d. random variables, each distributed as v. Going to the limit $t \to \infty$ in this formula, we see that the limit behavior of $\Delta_\sigma a_s^t$ depends on the convergence of the following random series:

$$\lim_{t\to\infty} \Delta_\sigma a_s^t = \sum_{n=0}^{\infty} \sum_s \Delta_\sigma p_n(s) \cdot v_s^{-n} \tag{10}$$

Lemma 1. The sum of $|\Delta_\sigma p_n(s)|^r$ over all $n=0, 1, 2, \ldots$ and $s \in \mathbf{Z}^d$ converges if and only if $r > (d+2)/(d+|\sigma|)$.

Proof. Lemma 1 is a direct corollary of the following formula (11). For every $r > 0$

$$\sum_s |\Delta_\sigma p_n(s)|^r \asymp (\sqrt{n})^{d-r(d+|\sigma|)} \qquad \text{when} \quad n \to \infty \tag{11}$$

To prove (11), let us introduce a d-dimensional random variable ω which equals $v_1, \ldots, v_n$ with probabilities $w_1, \ldots, w_n$. Notice that our coefficients $p_n(s)$, which were defined above, equal the following probabilities: $p_n(s) = \mathrm{Prob}(\omega_1 + \cdots + \omega_n = s)$, where $\omega_1, \ldots, \omega_n$ are i.i.d. variables, distributed as ω. This representation helps to obtain the following asymptotic expansion of $\Delta_\sigma p_n(s)$, which holds for all $\sigma \in \mathbf{Z}_+^d$ and $m \in \mathbf{Z}_+$ (here m is the number of terms in the expansion and the last term is the residue term):

$$\begin{aligned}\Delta_\sigma p_n(s) = \exp(-Q(x_n(s))) \sum_{k=0}^{m} \frac{P_k(x_n(s))}{(\sqrt{n})^{d+|\sigma|+k}} \\ + \frac{R_n(s)}{(1+\|x_n(s)\|^{m+2}) \cdot (\sqrt{n})^{d+|\sigma|+m}}\end{aligned} \tag{12}$$

where

$$x_n(s) = \frac{s - n\mu}{\sqrt{n}}, \qquad \lim_{n\to\infty} \sup_s |R_n(s)| = 0$$

μ is the mean of ω, $Q(x)$ is a positive-definite quadratic form, every $P_k(x)$ is a polynomial, and $P_0(x)$ is not identically equal to zero. Arguments of $Q(x)$ and $P_k(x)$ are components of x; coefficients of $Q(x)$ and $P_k(x)$ are constants. Equation (12) can be proved by induction based on the Cramér–Edgeworth asymptotic expansion for convolutions of identical lattice distributions, which is published in a form appropriate for us as Corollary 22.3 (p. 237) in Chapter 5 of ref. 2. Lemma 1 is proved. ∎

Now to prove Theorem 1.

Case $d=1$, $\sigma=0$. In this case $(d+2)/(d+|\sigma|)=3$. From Theorem 2 and Lemma 1, a_s^t certainly diverges.

Case $d=2$, $\sigma=(0,0)$. In this case $(d+2)/(d+|\sigma|)=2$. This is a boundary case, which is handled by statement (c) of Theorem 2. From Lemma 1 the sum of squares of p_s^n diverges, whence the series (4) also diverges.

In all the other cases $(d+2)/(d+|\sigma|)<2$. So, according to Theorem 2, the series (10) converges if $\text{Deg}(|\Delta_\sigma p_n(s)|) < \text{P-decay}(\nu)$ and diverges if $\text{Deg}(|\Delta_\sigma p_n(s)|) > \text{P-decay}(\nu)$. Thus statement (b) of Theorem 1 follows from Lemma 1.

3. PROOF OF THEOREM 2

Given two functions f and g, let $f \prec g$ mean that $f = O(g)$ and let $f \asymp g$ mean that $f \prec g$ and $g \prec f$. It is easy to prove that

$$\text{P-decay}(\xi) = \liminf_{x \to \infty} (-\log_x \text{Prob}(\xi > x)) \tag{13}$$

and that P-decay(ξ) equals the supremum of those r for which

$$\text{Prob}(\xi > x) \prec x^{-r} \qquad \text{when} \quad x \to \infty \tag{14}$$

When proving the statements (a) and (b) of Theorem 2 we use Kolmogorov's three-series theorem. In dealing with the series (4), all the summands of which are independent multiples of one and the same symmetric r.v. ξ, this theorem can be simplified as follows: the series (4) converges if and only if the following two series converge:

$$\sum_{k=1}^{\infty} \text{Prob}(\xi \geqslant 1/p_k) \tag{15}$$

and

$$\sum_{k=1}^{\infty} p_k^2 \int_0^{1/p_k} x^2 \, dF_\xi(x) \tag{16}$$

Proof of Statement (a). First assume that $\text{Deg}(p_k) > 2$. Then $\sum_{k=1}^{\infty} p_k^2$ diverges. Since the sequence p_k tends to zero and is nonincreasing, we can choose $k = k_0$ such that the integral in (16) exceeds a positive constant for all $k \geqslant k_0$. Therefore (16) diverges. ▮

Now assume that $\text{Deg}(p_k) > \text{P-decay}(\xi)$ and prove that the series (15) diverges. Let us estimate the sum (15) by an integral as follows:

$$\sum_{k=1}^{\infty} \text{Prob}(\xi \geqslant 1/p_k) \geqslant \int_0^{\infty} K(x) \, dF_\xi(x) - 1 \tag{17}$$

where $K(x)$ is defined as the smallest natural k for which $x \leqslant 1/p_k$. Let us choose a such that $\text{Deg}(p_k) > a > \text{P-decay}(\xi)$. From (3), $\liminf_{k \to \infty}(-\log_k p_k) < 1/a$, whence $1/p_k < k^{1/a}$ for k large enough. Hence $x^a \leqslant K(x)$ for x large enough. Hence, since the integral of x^a from 0 to ∞ diverges, the integral (17) also diverges.

Proof of Convergence in Statement (b). Choose $P < \text{P-decay}(\xi)$ and $R < 1/\text{Deg}(p_i)$ such that $R > 1/2$ and $P \cdot R > 1$. After that, due to (14), we can choose x_0 such that

$$\bar{F}(x) = \text{Prob}(\xi > x) \leqslant x^{-P} \qquad \text{for all} \quad x \geqslant x_0 \tag{18}$$

Also, from (3), we can choose k_0 such that $p_k \leqslant k^{-R}$ for all $k \geqslant k_0$.

Let us prove convergence of (15). From (18) for k large enough

$$\text{Prob}(\xi > 1/p_k) \leqslant (1/p_k)^{-P} = p_k^P \leqslant (k^{-R})^P = k^{-R \cdot P}$$

Since $R \cdot P > 1$, the sum of these terms converges. Hence (15) converges also.

Now prove convergence of (16). Denote $M = 1/p_k$ and $y = x^2$ and transform the integral in (16) using integration by parts:

$$\int_0^M x^2 \, dF_\xi(x) = \int_0^{M^2} (F(M) - F(\sqrt{y})) \, dy = \int_0^{M^2} (\bar{F}(\sqrt{y}) - \bar{F}(M)) \, dy$$

$$\leqslant \int_0^{M^2} \bar{F}(\sqrt{y}) \, dy = \int_0^{x_0^2} \bar{F}(\sqrt{y}) \, dy + \int_{x_0^2}^{M^2} \bar{F}(\sqrt{y}) \, dy$$

Here the first addend does not exceed x_0^2, which is a constant. Let us estimate the second one:

$$\int_{x_0^2}^{M^2} \bar{F}(\sqrt{y}) \, dy \leqslant \int_{x_0^2}^{M^2} \sqrt{y}^{-P} \, dy = \int_{x_0^2}^{M^2} y^{-P/2} \, dy$$

If $P > 2$, this integral does not exceed a constant, because we can substitute ∞ instead of M^2 as the upper limit, and the integral still converges. The case $P = 2$ can be avoided by choosing another P. So let $P < 2$. Then the last integral equals

$$\text{const} \cdot y^{1 - P/2} \big|_{x_0^2}^{M^2} \leqslant \text{const} \cdot p_k^{P-2}$$

Thus (16) does not exceed

$$\text{const} \cdot \sum_{k=1}^{\infty} p_k^2 (1 + p_k^{P-2}) \asymp \sum_{k=1}^{\infty} p_k^2 + \sum_{k=1}^{\infty} p_k^P$$

Since $p_k \leqslant k^{-R}$ for $k \geqslant k_0$, these series (several terms omitted) are majorized by

$$\sum_{k=k_0}^{\infty} k^{-2R} + \sum_{k=k_0}^{\infty} k^{-R \cdot P}$$

Both series converge due to our choice of P and R. Hence (16) converges. ▮

Proof of Divergence and Convergence in Statement (c). In one direction. Assume that $\sum p_k^2$ diverges. Since $p_k \to 0$, the integral

$$\int_0^{1/p_k} x^2 \, dF_\xi(x) \tag{19}$$

exceeds a positive constant for $k > \text{const}$. Therefore (16) diverges.

In the other direction. Assume that P-decay$(\xi) > 2$ and $\sum p_k^2$ converges and prove that the series (15) and (16) converge. The series (16) converges because the integral (19) does not exceed the integral of the same function from zero to infinity, which converges. Let us prove that (15) converges. Choose a such that $2 < a <$ P-decay(ξ). Then, due to (13),

$$a < \liminf_{x \to \infty} \left(-\log_x \operatorname{Prob}(\xi > x) \right)$$

whence there is x_0 such that $\operatorname{Prob}(\xi > x) < x^{-a}$ for all $x > x_0$. Therefore for k large enough the terms of (15) do not exceed p_k^a. The sum $\sum p_k^a$ converges, because $a > 2$ and $\sum p_k^2$ converges. ▮

Proof of Equality in Statements (b) and (c). It is easy to prove that

$$\text{P-decay}(\xi + \eta) = \min(\text{P-decay}(\xi), \text{P-decay}(\eta))$$

for any independent symmetric r.v. ξ and η. Hence

$$\text{P-decay}(\theta) = \text{P-decay}\left(p_1 \xi_1 + \sum_{k=2}^{\infty} p_k \xi_k \right) \leqslant \text{P-decay}(p_1 \xi_1) = \text{P-decay}(\xi)$$

In the other direction: assume that P-decay$(\theta) <$ P-decay(ξ) and come to a contradiction. Since $\operatorname{Deg}(p_k) <$ P-decay(ξ), we can choose r such that

$$\left. \begin{array}{l} \text{P-decay}(\theta) \\ \operatorname{Deg}(p_k) \end{array} \right\} < r < \text{P-decay}(\xi)$$

Now consider two cases.

Case $r<2$. In this case

$$\mathbf{E}\,|\theta|^r \leqslant 2\cdot\sum_{k=1}^{\infty}\mathbf{E}\,|p_k\xi_k|^r = 2\left(\sum_{k=1}^{\infty}p_k^r\right)\cdot\mathbf{E}\,|\xi|^r \tag{20}$$

This follows from the formula (2.29) on p. 58 of ref. 15 in the case $r\leqslant 1$ and from the von Bahr–Esseen inequality in the case $1<r<2$. (See, e.g., the formula at the bottom of p. 33 in ref. 3.) The sum on the right side of (20) converges because $\operatorname{Deg}(p_k)<r$ and $\mathbf{E}\,|\xi|^r$ is finite because $r<\text{P-decay}(\xi)$. Therefore $\mathbf{E}|\theta|^r$ is also finite, which contradicts our assumption.

Case $r\geqslant 2$. In this case it follows from Rosenthal's inequality (see e.g. p. 59 of ref. 15) that

$$\begin{aligned}(\mathbf{E}\,|\theta|^r)^{1/r} &\prec \left(\sum_{k=1}^{\infty}\mathbf{E}\,|p_k\xi_k|^r\right)^{1/r} + \left(\sum_{k=1}^{\infty}\mathbf{E}\,|p_k\xi_k|^2\right)^{1/2}\\ &= \left(\left(\sum_{k=1}^{\infty}p_k^r\right)\cdot\mathbf{E}\,|\xi|^r\right)^{1/r} + \left(\left(\sum_{k=1}^{\infty}p_k^2\right)\cdot\mathbf{E}\,|\xi|^2\right)^{1/2}\end{aligned}$$

Here the first term is finite because $\operatorname{Deg}(p_k)<r<\text{P-decay}(\xi)$. The second term is also finite because $\operatorname{Deg}(p_k)\leqslant 2<\text{P-decay}(\xi)$ and $\sum p_k^2$ converges. Therefore $\mathbf{E}\,|\theta|^r$ is also finite, which contradicts our assumption. ∎

4. PROOF OF THEOREM 4.

Convergence follows from Theorem 2, because $\text{P-decay}(\xi)$ is infinite whenever $\text{E-decay}(\xi)$ is positive. It remains to prove (7). Actually we shall prove the following inequalities, where $D=\operatorname{Deg}(p_k)$:

$$\text{E-decay}(\theta)\leqslant\text{E-decay}(\xi) \tag{21}$$

$$\text{E-decay}(\theta)\leqslant\frac{D}{D-1}\quad\text{if}\quad D>1 \tag{22}$$

$$\text{E-decay}(\theta)\geqslant\text{E-decay}(\xi)\quad\text{if}\quad D\leqslant 1\quad\text{or}\quad \text{E-decay}(\xi)\leqslant\frac{D}{D-1} \tag{23}$$

$$\text{E-decay}(\theta)\geqslant\frac{D}{D-1}\quad\text{if}\quad D>1\quad\text{and}\quad \text{E-decay}(\xi)>\frac{D}{D-1} \tag{24}$$

Proof of (21). It is easy to prove that

$$\text{E-decay}(\xi+\eta)=\min(\text{E-decay}(\xi),\,\text{E-decay}(\eta)) \tag{25}$$

for any independent symmetric r.v. ξ and η. Hence

$$\text{E-decay}(\theta) = \text{E-decay}\left(p_1\xi_1 + \sum_{k=2}^{\infty} p_k\xi_k\right) \leqslant \text{E-decay}(p_1\xi_1) = \text{E-decay}(\xi)$$

Proof of (22). Since ξ is symmetric and nonconstant, we can choose positive constants C and ε such that $\text{Prob}(\xi \geqslant C) = \varepsilon > 0$. Since $D > 1$, we can choose a and b such that $1/D < a < b < 1$. Then there is a sequence $k_1, k_2, \ldots \to \infty$ such that $p_{k_i} > k_i^{-a}$ for all i. Let E_n denote the following event:

$$\xi_k \geqslant C \qquad \text{for all} \quad k \in \{1, \ldots, n\}$$

Event E_{k_i} given,

$$\sum_{j=1}^{k_i} p_j\xi_j \geqslant C \cdot k_i \cdot k_i^{-a} = C \cdot k_i^{1-a}$$

Denote $x_i = C \cdot k_i^{1-a}$. The probability of E_{k_i} is not less than ε^{k_i}. The probability that $\sum_{j=k_i+1}^{\infty} p_j\xi_j \geqslant 0$ is not less than 1/2. Therefore

$$\text{Prob}(\theta \geqslant x_i) \geqslant \tfrac{1}{2} \cdot \varepsilon^{k_i}, \qquad \text{where} \quad k_i = (x_i/C)^{1/(1-a)}$$

Hence

$$\ln \text{Prob}(\theta \geqslant x_i) \geqslant \ln(1/2) + \ln \varepsilon \cdot (x_i/C)^{1/(1-a)}$$

Since $a < b < 1$, the last expression is greater than $-x_i^{1/(1-b)}$ for large enough i. Thus we have presented a sequence $x_1, x_2, \ldots \to \infty$ such that

$$\ln \text{Prob}(\theta \geqslant x_i) \geqslant -x_i^{1/(1-b)} \qquad \text{for all } i$$

Therefore $\text{E-decay}(\theta) \leqslant 1/(1-b)$. Since this is true for any b between $1/D$ and 1, (22) is proved.

Proof of (23). In the case $\text{E-decay}(\xi) \leqslant 1$ it follows from Theorem 2.3.2 on p. 41 in ref. 10. Let $\text{E-decay}(\xi) > 1$. Denote $\psi_\xi(z)$ the moments generation function, or MGF for short, of ξ:

$$\psi_\xi(z) = \psi(z \mid \xi) = \int_{-\infty}^{\infty} \exp(zx)\, dF_\xi(x)$$

Since $\text{E-decay}(\xi) > 1$, this integral converges.

Lemma 2. Take any symmetric r.v. ξ. Let E-decay$(\xi) > 1$. Then for any z_0 there is C such that $\ln \psi_\xi(z) \leqslant C \cdot z^2$ for all nonnegative $z \leqslant z_0$.

Proof. Since E-decay$(\xi) > 0$, P-decay$(\xi) = \infty$, whence all moments of ξ are finite, whence MGF has all derivatives at zero. Also note that MGF is even and equals one at $z = 0$. Hence $\psi_\xi(z) \leqslant 1 + \text{const} \cdot z^2$. Hence Lemma 2 follows. ▌

Given a symmetric r.v. ξ, let us call its *order* the following limit:

$$\text{Ord}(\xi) = \limsup_{z \to \infty} \log_z \ln \psi_\xi(z)$$

Since E-decay$(\xi) > 1$, Theorem 2.2.2 on p. 25 of ref. 12 ensures that the characteristic function of ξ is an entire function. What we call order is the same as what ref. 12 calls order and what we call E-decay is the same as what is denoted κ in ref. 12. This allows us to use the formula (2.4.3) on p. 37 of ref. 12 to conclude that

$$\frac{1}{\text{Ord}(\xi)} + \frac{1}{\text{E-decay}(\xi)} = 1 \tag{26}$$

Due to (26), it is sufficient to prove that $\text{Ord}(\theta) \leqslant \text{Ord}(\xi)$. Take some $a > \text{Ord}(\xi)$ and prove that $\text{Ord}(\theta) \leqslant a$. There is z_0 such that $\ln \psi_\xi(z) \leqslant z^a$ for all $z \geqslant z_0$. After that, due to Lemma 2, we can choose C such that $\ln \psi_\xi(z) \leqslant C \cdot z^2$ for all nonnegative $z \leqslant z_0$. Now consider two cases.

Case 1. E-decay$(\xi) \leqslant 2$. Remember that the MGF of a convolution of several distributions equals the product of their MGFs. Hence

$$\psi_\theta(z) = \prod_{k=1}^{\infty} \psi(z \mid p_k \cdot \xi_k) = \prod_{k=1}^{\infty} \psi_\xi(p_k \cdot z)$$

where θ is defined in (6). Using this and Lemma 2, we can write

$$\begin{aligned} \ln \psi_\theta(z) &= \sum_{k=1}^{\infty} \ln \psi_\xi(p_k \cdot z) \\ &\leqslant \sum_{k=1}^{\infty} (C \cdot (p_k \cdot z)^2 + (p_k \cdot z)^a) = z^2 \cdot C \sum_{k=1}^{\infty} p_k^2 + z^a \cdot \sum_{k=1}^{\infty} p_k^a \end{aligned} \tag{27}$$

Since E-decay$(\xi) \leqslant 2$, (26) implies $\text{Ord}(\xi) \geqslant 2$, whence $a > 2$. So both series (27) converge. Since $z \to \infty$, we may neglect the first addend in (27) and write $\ln \psi_\theta(z) \prec z^a$. Hence $\text{Ord}(\theta) \leqslant a$. ▌

Case 2. $2 < \text{E-decay}(\xi) \leqslant \infty$. Since $\text{E-decay}(\xi) > 2$, (26) implies $\text{Ord}(\xi) < 2$. So we may choose a such that $\text{Ord}(\xi) < a \leqslant 2$. Then $z^2 \leqslant z^a$ for $0 \leqslant z \leqslant 1$. Therefore $\ln \psi_\xi(z) \prec z^a$ for all z, whence

$$\ln \psi_\theta(z) = \sum_{k=1}^{\infty} \ln \psi_\xi(p_k \cdot z) \prec z^a \cdot \sum_{k=1}^{\infty} p_k^a \tag{28}$$

Since $\text{E-decay}(\xi) \leqslant D/(D-1)$, (26) implies $\text{Ord}(\xi) \geqslant \text{Deg}(p_k)$, whence $a > \text{Deg}(p_k)$, whence the series (28) converges. Therefore $\ln \psi_\theta(z) \prec z^a$, whence $\text{Ord}(\theta) \leqslant a$. ∎

Proof of (24). Due to (26), all we need to prove is $\text{Ord}(\theta) \leqslant \text{Deg}(p_k)$. Since $\text{Deg}(p_k) < 2$, it is sufficient to choose any b between them and prove that $\text{Ord}(\theta) \leqslant b$. The assumption $\text{E-decay}(\xi) > D/(D-1)$ implies $\text{Ord}(\xi) < \text{Deg}(p_k)$. So we can choose some a between them. Thus $\text{Ord}(\xi) < a < \text{Deg}(p_k) < b < 2$. Since $\text{Ord}(z) < a$, there is z_0 such that $\ln \psi_\xi(z) \leqslant z^a$ for all $z \geqslant z_0$. After that, due to Lemma 2, we can choose C such that $\ln \psi_\xi(z) \leqslant C \cdot z^2$ for $0 \leqslant z \leqslant z_0$. Also remember that we may assume that the sequence p_k is nonincreasing. So we can write

$$\begin{aligned}\ln \psi_\theta(z) &= \sum_{k=1}^{\infty} \ln \psi_\xi(p_k \cdot z) = \sum_{k=1}^{v-1} \ln \psi_\xi(p_k \cdot z) + \sum_{k=v}^{\infty} \ln \psi_\xi(p_k \cdot z)\\ &\prec \sum_{k=1}^{v-1} (p_k \cdot z)^a + \sum_{k=v}^{\infty} (p_k \cdot z)^2 = z^a \cdot \sum_{k=1}^{v-1} p_k^a + z^a \cdot \sum_{k=v}^{\infty} p_k^2 \end{aligned}\tag{29}$$

where v is such that $p_{v-1} \geqslant z_0/z \geqslant p_v$. Since $\text{Deg}(p_k) < b$, we can find k_0 such that $p_k \leqslant k^{-1/b}$ for $k \geqslant k_0$. Since $z \to \infty$, $v \to \infty$ also, so we may assume that $v > k_0$. Therefore $z_0/z \leqslant p_v \leqslant v^{-1/b}$, whence $v \leqslant (z/z_0)^b$. Using this, we estimate the first sum in (29):

$$\sum_{k=1}^{v-1} p_k^a = \sum_{k=1}^{k_0-1} p_k^a + \sum_{k=k_0}^{v-1} p_k^a \leqslant \text{const} + \sum_{k=k_0}^{v-1} k^{-a/b} \prec v^{1-(a/b)} \prec z^{b-a}$$

whence the first summand in (29) does not exceed $\text{const} \cdot z^b$. Now denote $w = [(z/z_0)^b]$ and split the second summand in (29) into two parts:

$$z^2 \cdot \sum_{k=v}^{\infty} p_k^2 = z^2 \cdot \sum_{k=v}^{w} p_k^2 + z^2 \cdot \sum_{k=w+1}^{\infty} p_k^2$$

Here the first summand does not exceed

$$z^2 \cdot \sum_{k=v}^{w} \left(\frac{z_0}{z}\right)^2 \prec w \prec z^b$$

The second summand does not exceed

$$z^2 \cdot \sum_{k=w+1}^{\infty} k^{-2/b} \prec z^2 \cdot w^{1-(2/b)} \prec z^b$$

Thus $\ln \psi_\theta(z) \prec z^b$ when $z \to \infty$. Therefore $\mathrm{Ord}(\theta) \leqslant b$. ▮

5. EXAMPLES AND NOTES

Example 1. Consider the ν-dimensional one-sided harness as defined in formula (3.3) of ref. 6 (using notations which are more convenient here):

$$a(r_1, r_2,..., r_\nu) = \frac{1}{\nu}(a(r_1 - 1, r_2,..., r_\nu) + \cdots + a(r_1, r_2,..., r_\nu - 1)) + \xi(r_1,..., r_\nu)$$

where ξ is the random noise. The initial condition is

$$a(0, r_2, r_3,..., r_\nu) = \cdots = a(r_1, r_2,..., r_{\nu-1}, 0) = 0$$

and the role of time is played by the sum $r_1 + \cdots + r_d$. By an affine linear transformation of the space-time continuum this model can be turned into a special case of (1) with $d = \nu - 1$, $N = d + 1$, $v_1,..., v_{d+1}$ being vertices of a simplex, and all $w_1,..., w_{d+1}$ being equal to each other (and therefore equal to $1/(d+1)$). For example, we may take $v_1 = e_1,..., v_d = e_d$, $v_{d+1} = 0$, where $e_1,..., e_d$ are the orts. The initial condition remains different from ours, but our random series approach shows that this difference is unimportant for convergence.

Example 2. Consider the Edwards–Wilkinson equation in the form of (4.1) on p. 135 of ref. 5

$$\frac{\partial h}{\partial t} = \nu\, \nabla^2 h + \eta(\mathbf{r}, t)$$

where h is the height of a sandpile. The first term on the right side represents the surface relaxation, ν being the diffusion coefficient. The second term $\eta(\mathbf{r}, t)$ is the random noise. This equation is interesting also as a linearized version of the KPZ equation.[7] If we discretize this equation, $\partial h/\partial t$ turns into $h(\mathbf{r}, t+1) - h(\mathbf{r}, t)$, $\nabla^2 h$ turns into a linear combination of $h(\mathbf{r} + v_i, t)$, where $\mathbf{r} + v_i$ are several neighbors of $\mathbf{r}$, and the whole equation turns into a special case of (1).

Example 3. Take ξ distributed as

$$\frac{dF_\xi(x)}{dx}=\frac{\text{const}}{1+|x|^{r+1}} \tag{30}$$

with $r>2$. Take

$$p_k=\frac{1}{\sqrt{k}} \qquad \text{or} \qquad p_k=\frac{1}{\sqrt{k}\cdot\ln(k+e)}$$

The series $\sum p_k^2$ diverges in the former case and converges in the latter. Therefore the series $\sum p_k\xi_k$ diverges in the former case and converges in the latter. This example shows that both divergence and convergence of (4) are possible for all values of P-decay$(\xi)>\text{Deg}(p_k)=2$.

Example 4. Take ξ defined by (30) with $0<r<2$. Take

$$p_k=k^{-1/r} \qquad \text{or} \qquad p_k=p^{-1/r}\cdot\ln^{-2/r}(k+e)$$

In the former case both (15) and (16) diverge, in the latter case both converge. This example shows that both divergence and convergence of (4) are possible for all values of P-decay(ξ) and Deg(p_k) in the range $0<\text{P-decay}(\xi)=\text{Deg}(p_k)<2$.

Note 1. Following ref. 6, we might assume that v has a variance, and ask how the variances of $\Delta_\sigma a_s^t$ behave when $t\to\infty$. From (9)

$$\text{Var}(\Delta_\sigma a_0^t)=\text{Var}(v)\cdot\sum_{n=0}^{t-1}\sum_s(\Delta_\sigma p_n(s))^2$$

and it remains to examine how this sum behaves when $t\to\infty$. From (11)

$$\sum_s(\Delta_\sigma p_n(x))^2\asymp(\sqrt{n})^{-d-2|\sigma|}$$

A series with these terms converges if and only if $d+2|\sigma|>2$, that is, in all cases except those mentioned in Theorem 1. In more detail:

If $d=1$ and $\sigma=0$, $$\sum_{n=0}^{t-1}\sum_s(p_n(s))^2\asymp\sum_{n=0}^{t-1}\frac{1}{\sqrt{n}}\asymp\sqrt{t}$$

If $d=2$ and $\sigma=(0,0)$, $$\sum_{n=0}^{t-1}\sum_s(p_n(s))^2\asymp\sum_{n=0}^{t-1}\frac{1}{n}\asymp\log t$$

In all the other cases $\mathrm{Var}(\lim_{t \to \infty} \Delta_\sigma a_s^t)$ is finite. Thus we repeat some of the results of ref. 6 by other means.

Example 5. Take ξ distributed as

$$\frac{dF_\xi(x)}{dx} = \mathrm{const} \cdot \exp(-|x|^r), \qquad \text{where} \quad r > 0$$

Take

$$p_k = \frac{1}{\sqrt{k}} \qquad \text{or} \quad p_k = \frac{1}{\sqrt{k} \cdot \ln^2(k+e)}$$

The series (6) diverges in the former case and converges in the latter. Thus both convergence and divergence of (6) are possible for every value of E-decay(ξ) if $\mathrm{Deg}(p_k) = 2$.

Note 2. In both Theorems 3 and 4 we excluded the case when E-decay of ν, resp. ξ, is zero. However, all the assertions of our theorems are true in this case also as soon as convergence takes place.

Note 3. Another case which we excluded from Theorem 4 is when $\mathrm{Deg}(p_k) = 2$, but the series (6) still converges, because the series $\sum p_k^2$ converges. In this case the formula (7) is also true. To check this, one can review the arguments and see that whenever we use the condition $\mathrm{Deg}(p_k) < 2$, all we actually need is convergence of $\sum p_k^2$. Only when proving (24) did we have to be careful. Instead of assuming $\mathrm{Deg}(p_k) < b < 2$, we should assume $\mathrm{Deg}(p_k) = 2 < b$ and when estimating the last addend, we should refer to convergence of $\sum p_k^2$.

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